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Negaton and positon solutions of the soliton equation with self-consistent sources

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Abstract

The Korteweg–de Vries (KdV) equation with self-consistent sources (KdVES) is used as a model to illustrate this method. We present a generalized binary Darboux transformation (GBDT) with an arbitrary time-dependent function for the KdVES as well as the formula for N-times repeated GBDT. This GBDT provides non-auto-Bäcklund transformation between two KdV equations with different degrees of sources and enables us to construct more general solutions with N arbitrary t-dependent functions. By taking the special t-function, we obtain multisoliton, multipositon, multinegaton, multisoliton–positon, multinegaton–positon and multisoliton–negaton solutions of the KdVES.

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1. Introduction

Soliton equations with self-consistent sources (SESCS) have attracted some attention (see, for example, [1–14]). The SESCS can be solved by the inverse scattering method and N-soliton solutions of some SESCSs have been obtained [1–15]. However, since the explicit time part of the Lax representation for SESCS was not found, the determination of the evolution for scattering data was quite complicated in [1–14]. In recent years, we have presented the time part of the Lax representation for the SESCS by means of the adjoint representation of the soliton equation [16, 17]. This enables us to determine the evolution of scattering data in a simple and natural way [15] and to construct the Darboux transformation for the SESCS [18, 19]. It has been pointed out in [18, 19] that the normal Darboux transformation for the SESCS, which provides auto-Bäcklund transformation, cannot be used to construct a solution of the SESCS from the trivial solution. In [18, 19] we have presented a special type of binary Darboux transformation for some SESCSs, which offers non-auto-Bäcklund transformation between soliton equations with different degrees of sources and can be used to obtain N-soliton solutions. To our knowledge, no other solutions, except the soliton solution, for the SESCS have been investigated.

In recent years, positon and negaton solutions of soliton equations have been widely studied (see [20] and references therein). The positon solutions of soliton equations are long-range analogues of solitons and slowly decreasing, oscillating solutions, and possess a so-called supertransparent property; the corresponding reflection coefficient is zero and the transmission coefficient is unity [20]. The negaton solution of the Korteweg–de Vries (KdV) equation has been studied in [21].

In this paper, we use the KdV equation with self-consistent sources (KdVES) as a model to illustrate this idea. We present generalized binary Darboux transformation (GBDT) with arbitrary t-dependent functions for the KdVES and the formula for N-times repeated GBDT which contains N arbitrary t-dependent functions. This GBDT offers a non-auto-Bäcklund transformation between KdV equations with different degrees of sources and enables us to find the more general solution with arbitrary t-functions for the KdVES. By taking the special t-function, we obtain multisoliton, multipositon, multinegaton, multisoliton–positon, multisoliton–negaton and multipositon–negaton solutions of the KdVES.

This paper is organized as follows. In section 2, we derive the GBDT with an arbitrary t-dependent function for the KdVES and the formula for N-times repeated GBDT with an N arbitrary t-dependent function. Using this GBDT gives rise to some general solutions of the KdVES including the multisoliton solution as a special case. In sections 3 and 4, multipositon and multinegaton solutions of the KdVES are obtained, respectively. Finally, in section 5, multisoliton–positon, multisoliton–negaton and multipositon–negaton solutions of the KdVES are presented.

2. The generalized binary Darboux transformation

The KdV equation with sources of degree n (KdVES) is defined by [3, 5, 14, 15, 18]

$$u_t + 6uu_x + u_{xxx} + 4\sum_{j=1}^{n} \varphi_j \varphi_{j,x} = 0 \tag{2.1a}$$

$$\varphi_{j,xx} + (\lambda_j + u)\varphi_j = 0 \qquad j = 1, \ldots, n \tag{2.1b}$$

where λ_j are distinct real constants. Let $\Phi_n = (\varphi_1, \ldots, \varphi_n)$. The Lax representation for equation (2.1) can be found from the adjoint representation for the KdV equation [16–18]

$$\phi_{xx} + (\lambda + u)\phi = 0 \tag{2.2a}$$

$$\phi_t = A_n(\lambda, u, \Phi_n)\phi \tag{2.2b}$$

where

$$A_n(\lambda, u, \Phi_n)\phi = u_x\phi + (4\lambda - 2u)\phi_x + \sum_{j=1}^{n} \frac{\varphi_j}{\lambda_j - \lambda} W(\varphi_j, \phi)$$

and $W(\varphi_j, \phi) \equiv \varphi_j\phi_x - \varphi_{j,x}\phi$ is the usual Wronskian determinant. It is shown that the well-known Darboux transformation (DT) for the KdV equation can be applied to the KdVES [18]. Let f be a solution of equation (2.2) with $\lambda = \xi$, then equation (2.2) is covariant under the DT defined as [18]

$$\widetilde{\phi} = \frac{W(f, \phi)}{f} \tag{2.3a}$$

$$\widetilde{u} = u + 2\partial_x^2 \ln f \tag{2.3b}$$

$$\widetilde{\varphi}_j = \frac{1}{\sqrt{\lambda_j - \xi}} \frac{W(f, \varphi_j)}{f} \qquad j = 1, \ldots, n \tag{2.3c}$$

i.e., $\widetilde{\phi}$, $\widetilde{u}$ and $\widetilde{\Phi}_n = (\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_n)$ satisfy

$$\widetilde{\phi}_{xx} + (\lambda + \widetilde{u})\widetilde{\phi} = 0 \tag{2.4a}$$

$$\widetilde{\phi}_t = A_n(\lambda, \widetilde{u}, \widetilde{\Phi}_n)\widetilde{\phi} \tag{2.4b}$$

and $\widetilde{u}$, $\widetilde{\Phi}_n$ is a new solution of equation (2.1). Through this DT, we can find two linearly independent solutions of equation (2.4) with $\lambda = \xi$. First, equation (2.3*a*) gives a solution of equation (2.4) with $\lambda = \xi$

$$\widetilde{f}_1 = \frac{C}{f} \tag{2.5}$$

where C is some constant. Secondly, letting g be a solution of equation (2.2) with $\lambda = \eta \neq \xi$, we define

$$\omega(f, g) = \frac{W(f, g)}{\xi - \eta} \qquad \omega(f, f) \equiv \lim_{\eta \to \xi} \frac{W(f(\xi), f(\eta))}{\xi - \eta} = -W(f, \partial_\xi f).$$

According to equation (2.3*a*)

$$\widetilde{f} = \lim_{\eta \to \xi} \frac{W(f(\xi), f(\eta))}{(\xi - \eta) f(\xi)} = \frac{1}{f}\omega(f, f)$$

is another solution of equation (2.4) with $\lambda = \xi$. Therefore

$$\widetilde{h} \equiv \widetilde{f} + \widetilde{f}_1 = \frac{1}{f}[C + \omega(f, f)] \tag{2.6}$$

is also a solution of equation (2.4) with $\lambda = \xi$. Using f and $\widetilde{h}$ consecutively, the two-times action of DT (2.3) yields the following binary DT

$$\bar{\phi} = \frac{1}{\lambda - \xi} \frac{W(\widetilde{h}, \widetilde{\phi})}{\widetilde{h}} = \phi - \frac{f}{C + \omega(f, f)} \omega(f, \phi) \tag{2.7a}$$

$$\bar{u} = \widetilde{u} + 2\partial_x^2 \ln \widetilde{h} = u + 2\partial_x^2 \ln[C + \omega(f, f)] \tag{2.7b}$$

$$\bar{\varphi}_j = \frac{1}{\sqrt{\lambda_j - \xi}} \frac{W(\widetilde{h}, \widetilde{\varphi}_j)}{\widetilde{h}} = \varphi_j - \frac{f}{C + \omega(f, f)} \omega(f, \varphi_j) \qquad j = 1, \ldots, n. \tag{2.7c}$$

Then, the system (2.2) is covariant under the binary DT (2.7) and $\bar{u}$, $\bar{\Phi}_n \equiv (\bar{\varphi}_1, \ldots, \bar{\varphi}_n)$ satisfies the KdVES (2.1).

Substituting equation (2.7*a*) into equation (2.2*b*) gives

$$\bar{\phi}_t = \phi_t - \frac{f_t \omega(f, \phi)}{C + \omega(f, f)} + \frac{f \partial_t \omega(f, f)}{[C + \omega(f, f)]^2} \omega(f, \phi) - \frac{f \partial_t \omega(f, \phi)}{C + \omega(f, f)} = A_n(\lambda, \bar{u}, \bar{\Phi}_n)\bar{\phi}. \tag{2.8}$$

When substituting equation (2.7) into $A_n(\lambda, \bar{u}, \bar{\Phi}_n)\bar{\phi}$, the last equality holds for any constant C. In the expression of $A_n(\lambda, u, \Phi_n)\phi$, there is no derivative with respect to t. So the last equality holds when C is replaced by $e(t)$, an arbitrary t-function. We have the following lemma.

Lemma 2.1. *Given* u, Φ_n *a solution of equation (2.1), if* f *is a solution of equation (2.2) with* $\lambda = \xi$, *then the last equality of equation (2.8) holds for* $C = e(t)$.

Obviously, under DT defined by equation (2.7) with C replaced by $e(t)$, equation (2.2*a*) is still covariant; however, equation (2.2*b*) is no longer covariant. In fact, we have

Theorem 2.1. *Given* u, Φ_n *as a solution of equation (2.1), let* f *be a solution of the system (2.2) with* $\lambda = \lambda_{n+1}$. *Then, the generalized binary DT with an arbitrary t-function defined by*

$$\bar{\phi} = \phi - \frac{f}{e(t) + \omega(f, f)}\omega(f, \phi) \tag{2.9a}$$

$$\bar{u} = u + 2\partial_x^2 \ln[e(t) + \omega(f, f)] \tag{2.9b}$$

$$\bar{\varphi}_j = \varphi_j - \frac{f}{e(t) + \omega(f, f)}\omega(f, \varphi_j) \qquad j = 1, \ldots, n \tag{2.9c}$$

and

$$\bar{\varphi}_{n+1} = \frac{\sqrt{e'(t)}f}{e(t) + \omega(f, f)} \tag{2.9d}$$

transforms equation (2.2) into

$$\bar{\phi}_{xx} + (\lambda + \bar{u})\bar{\phi} = 0 \tag{2.10a}$$

$$\bar{\phi}_t = A_{n+1}(\lambda, \bar{u}, \bar{\Phi}_{n+1})\bar{\phi} \tag{2.10b}$$

and $\bar{u}$, $\bar{\Phi}_{n+1} \equiv (\bar{\varphi}_1, \ldots, \bar{\varphi}_{n+1})$, *satisfy the KdV equation with sources of degree* $n + 1$

$$\bar{u}_t + 6\bar{u}\bar{u}_x + \bar{u}_{xxx} + 4\sum_{j=1}^{n+1} \bar{\varphi}_j\bar{\varphi}_{j,x} = 0 \tag{2.11a}$$

$$\bar{\varphi}_{j,xx} + (\lambda_j + \bar{u})\bar{\varphi}_j = 0 \qquad j = 1, \ldots, n + 1 \tag{2.11b}$$

Proof. $\widetilde{h}$ defined by equation (2.6) with C replaced by $e(t)$ still satisfies equation (2.4*a*). This implies that equations (2.10*a*) and (2.11*b*) hold. Substituting equation (2.9*a*) into the left-hand side of equation (2.10*b*) and using lemma 2.1 and equation (2.9*d*) gives rise to

$$\begin{aligned}\bar{\phi}_t = A_n(\lambda, \bar{u}, \bar{\Phi}_n)\bar{\phi} + \frac{e'(t)\omega(f, \phi)f}{[e(t) + \omega(f, f)]^2} &= A_n(\lambda, \bar{u}, \bar{\Phi}_n)\bar{\phi} \\ &+ \frac{W(\bar{\varphi}_{n+1}, \bar{\phi})}{\lambda_{n+1} - \lambda}\bar{\varphi}_{n+1} = A_{n+1}(\lambda, \bar{u}, \bar{\Phi}_{n+1})\bar{\phi}.\end{aligned}$$

Then the compatibility condition of equation (2.10) leads to equation (2.11*a*). This completes the proof.

The GBDT defined by equation (2.9) contains an arbitrary t-function. The flexibility of the choices of $e(t)$ and f enables us to construct some general solutions with arbitrary t-functions of the KdVES, some of which cannot be constructed through the original binary DT.

For m solutions of equation (2.2), $g_1, \ldots, g_m$ and m arbitrary t-functions $e_1(t), \ldots, e_m(t)$, we define two types of Wronskian determinant

$$W_1(g_1, \ldots, g_m; e_1, \ldots, e_m) = \det F \qquad W_2(g_1, \ldots, g_m; e_1, \ldots, e_{m-1}) = \det G$$

where

$$F_{ij} = \delta_{ij} e_i(t) + \omega(g_i, g_j) \qquad i, j = 1, \ldots, m$$
$$G_{ij} = \delta_{ij} e_i(t) + \omega(g_i, g_j) \qquad i = 1, \ldots, m-1 \qquad j = 1, \ldots, m$$
$$G_{mj} = g_j \qquad j = 1, \ldots, m.$$

We have the following formula of N-times repeated GBDT. □

Theorem 2.2. *Given u, Φ_n as a solution of equation (2.1), let $f_1, \ldots, f_N$ be solutions of equation (2.2) with $\lambda = \lambda_{n+1}, \ldots, \lambda_{n+N}$, respectively. Then the N-times repeated GBDT with N-arbitrary t-functions $e_1(t), \ldots, e_N(t)$ defined by*

$$\bar{\phi} = \frac{W_2(f_1, \ldots, f_N, \phi; e_1, \ldots, e_N)}{W_1(f_1, \ldots, f_N; e_1, \ldots, e_N)} \tag{2.12a}$$

$$\bar{u} = u + 2\partial_x^2 \ln W_1(f_1, \ldots, f_N; e_1, \ldots, e_N) \tag{2.12b}$$

$$\bar{\varphi}_j = \frac{W_2(f_1, \ldots, f_N, \varphi_j; e_1, \ldots, e_N)}{W_1(f_1, \ldots, f_N; e_1, \ldots, e_N)} \qquad j = 1, \ldots, n \tag{2.12c}$$

and

$$\bar{\varphi}_{n+j} = \frac{\sqrt{e_j'(t)}\, W_2(f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_N, f_j; e_1, \ldots, e_{j-1}, e_{j+1}, \ldots, e_N)}{W_1(f_1, \ldots, f_N; e_1, \ldots, e_N)} \qquad j = 1, \ldots, N \tag{2.12d}$$

transforms equation (2.2) into equation (2.2) with n replaced by $n + N$ and $\bar{u}, \overline{\Phi}_{n+N}$ satisfy the KdVES of degree $n + N$, i.e. equation (2.1) with n replaced by $n + N$.

The proof of this theorem is completely similar to that given in [18] and we omit it.

Example. N-soliton solution.

We take $u = 0$ as the initial solution of equation (2.1) with $n = 0$ and let $\lambda_j = -\kappa_j^2 < 0$, $\kappa_j > 0$, $j = 1, \ldots, N$,

$$f_j = \mathrm{e}^{\kappa_j x - 4\kappa_j^3 t} \qquad e_j(t) = \mathrm{e}^{2\alpha_j t} \qquad j = 1, \ldots, N$$

then

$$\omega(f_i, f_j) = \frac{1}{\kappa_i + \kappa_j} \mathrm{e}^{(\kappa_i + \kappa_j)x - 4(\kappa_i^3 + \kappa_j^3)t} \qquad i, j = 1, \ldots, N.$$

The N-soliton solutions of equation (2.1) with $n = N$ and $\lambda_j = -\kappa_j^2 < 0$, $j = 1, \ldots, N$, is given by

$$u = 2\partial_x^2 \ln W_1(f_1, \ldots, f_N; e_1, \ldots, e_N) \tag{2.13a}$$

$$\varphi_j = \frac{\sqrt{e_j'(t)}\, W_2(f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_N, f_j; e_1, \ldots, e_{j-1}, e_{j+1}, \ldots, e_N)}{W_1(f_1, \ldots, f_N; e_1, \ldots, e_N)} \qquad j = 1, \ldots, N \tag{2.13b}$$

which was obtained in [14, 15, 18].

3. Positon solutions

Hereafter we always take the simple and special choice of $e(t)$ as

$$e_j(t) = a_j t + b_j \tag{3.1}$$

where $a_j \neq 0$ and b_j are real constants.

3.1. One-positon solution and the supertransparency

We take $u = 0$ as the initial solution of equation (2.1) with $n = 0$. Let f be an oscillating solution of equation (2.2) with $u = 0, n = 0$ and $\lambda = \lambda_1 = \kappa^2 > 0, \kappa > 0$,

$$f = \sin\Theta \qquad \Theta = \kappa(x + x_1 + 4\kappa^2 t) \tag{3.2}$$

where $x_1 = x_1(\kappa)$ is a real differential function of κ. Then the GBDT (2.9) gives

$$u = 2\partial_x^2 \ln(2\kappa\gamma - \sin 2\Theta) = \frac{32\kappa^2 \sin\Theta(\kappa\gamma\cos\Theta - \sin\Theta)}{(2\kappa\gamma - \sin 2\Theta)^2} \tag{3.3a}$$

$$\varphi_1 = \frac{4\kappa\sqrt{a}\sin\Theta}{2\kappa\gamma - \sin 2\Theta} \tag{3.3b}$$

with

$$\gamma = \partial_\kappa\Theta + 2e(t) = x + \tilde{x}_1 + (12\kappa^2 + 2a)t + 2b \qquad \tilde{x}_1 = x_1 + \kappa\partial_\kappa x_1(\kappa)$$

which gives the one-positon solution of the KdVES (2.1) with $n = 1, \lambda_1 = \kappa^2$ corresponding to the one-positon solution for the KdV equation in [20, 21].

Based on formulae (3.3), we can analyse the basic features of the one-positon solution of equation (2.1) in the same way as in [20]. We can conclude that the one-positon solution of equation (2.1) with $n = 1$ has the same shape, the same asymptotic behaviour when $x \to \pm\infty$ and the same scattering data as the one-positon solution of the KdV equation, i.e. long-range analogues of solitons of the KdVES and slowly decreasing, oscillating solutions. Similarly, under a proper choice of scattering data, the corresponding reflection coefficient is zero and the transmission coefficient is unity.

3.2. Two-positon solution and multipositon solutions

The two-positon solution of equation (2.1) with $n = 2, \lambda_j = \kappa_j^2 > 0, \kappa_j > 0, j = 1, 2$, is given by equation (2.13) with $N = 2, e_j = a_j t + b_j$

$$f_j = \sin\Theta_j \qquad \Theta_j = \kappa_j\left(x + x_j + 4\kappa_j^2 t\right) \qquad \operatorname{Im} x_j = 0 \qquad j = 1, 2$$

$$\begin{aligned} W_1(f_1, f_2; e_1, e_2) = {} & (16\kappa_1\kappa_2)^{-1}(2\kappa_1\gamma_1 - \sin 2\Theta_1)(2\kappa_2\gamma_2 - \sin 2\Theta_2) \\ & - \left(\kappa_1^2 - \kappa_2^2\right)^{-2}(\kappa_2 \sin\Theta_1\cos\Theta_2 - \kappa_1\sin\Theta_2\cos\Theta_1)^2 \end{aligned} \tag{3.4a}$$

$$\begin{aligned} W_2(f_2, f_1; e_2) = {} & (4\kappa_2)^{-1}\sin\Theta_1(2\kappa_2\gamma_2 - \sin 2\Theta_2) \\ & - \left(\kappa_1^2 - \kappa_2^2\right)^{-1}\sin\Theta_2(\kappa_2\sin\Theta_1\cos\Theta_2 - \kappa_1\sin\Theta_2\cos\Theta_1) \end{aligned} \tag{3.4b}$$

$$\begin{aligned} W_2(f_1, f_2; e_1) = {} & (4\kappa_1)^{-1}\sin\Theta_2(2\kappa_1\gamma_1 - \sin 2\Theta_1) \\ & - \left(\kappa_1^2 - \kappa_2^2\right)^{-1}\sin\Theta_1(\kappa_2\sin\Theta_1\cos\Theta_2 - \kappa_1\sin\Theta_2\cos\Theta_1) \end{aligned} \tag{3.4c}$$

$$\gamma_j = x + \widetilde{x}_j + \left(12\kappa_j^2 + 2a_j\right)t + 2b_j \qquad \widetilde{x}_j = x_j + \kappa_j\partial_{\kappa_j}x_j(\kappa_j) \qquad j = 1, 2.$$

Using equation (3.4), we obtain the asymptotic behaviour of the solution for fixed γ_1 as $t \to \pm\infty$ (which implies $\gamma_2 \to \infty$)

$$u = 2\partial_x^2 \ln(2\kappa_1\gamma_1 - \sin 2\Theta_1)\left[1 + O\left(\gamma_2^{-1}\right)\right]$$
$$\varphi_1 = \frac{4\kappa_1\sqrt{a_1}\sin\Theta_1}{2\kappa_1\gamma_1 - \sin 2\Theta_1}\left[1 + O\left(\gamma_2^{-1}\right)\right] \qquad \varphi_2 = O\left(\gamma_2^{-1}\right).$$

When γ_2 is fixed and $t \to \pm\infty(\gamma_1 \to \infty)$, we have a similar result for the asymptotic behaviour of the solution. Thus, we have proven that the two positons are totally insensitive to the mutual collision, even without additional phase shifts, which is intrinsic for the collision of two solitons. Calculating the corresponding solution of system (2.2), we can prove that potential is also supertransparent.

The N-positon solution of equation (2.1) with $n = N, \lambda_j = \kappa_j^2 > 0, \kappa_j > 0, j = 1, \ldots, N$, is given by equation (2.13) with $e_j = a_j t + b_j$,

$$f_j = \sin\Theta_j \qquad \Theta_j = \kappa_j\left(x + x_j + 4\kappa_j^2 t\right) \qquad \operatorname{Im} x_j = 0 \qquad j = 1, \ldots, N.$$

Analogously, we see that the N-positon solution at large time decays into the sum of N free positons and it is also supertransparent.

4. Negaton solutions

4.1. One-negaton solution

Let $\lambda_1 = -\kappa^2 < 0, \kappa > 0$ and f be a solution of equation (2.2) with $u = 0, n = 0$ and $\lambda = \lambda_1$,

$$f = \sinh\Theta \qquad \Theta = \kappa(x + x_1 - 4\kappa^2 t). \tag{4.1}$$

Then the GBDT (2.9*b*) and (2.9*d*) with $e(t) = at + b$ gives

$$u = 2\partial_x^2 \ln(\kappa\gamma - \sinh\Theta\cosh\Theta) = \frac{8\kappa^2 \sinh\Theta(\sinh\Theta - \kappa\gamma\cosh\Theta)}{(\kappa\gamma - \sinh\Theta\cosh\Theta)^2} \tag{4.2a}$$

$$\varphi_1 = \frac{2\kappa\sqrt{a}\sinh\Theta}{\kappa\gamma - \sinh\Theta\cosh\Theta} \tag{4.2b}$$

where

$$\gamma = x + x_1 + \kappa\partial_\kappa x_1 - (12\kappa^2 - 2a)t + 2b. \tag{4.2c}$$

Equation (4.2) gives the [S] one-negaton solution of equation (2.1) with $n = 1$ and $\lambda_1 = -\kappa^2 < 0$, which corresponds to the [S] one-negaton solution for the KdV equation in [21].

When t is fixed, then we have

$$u \sim 8\kappa^2\left(\frac{1}{\cosh^2\Theta} - \frac{\kappa\gamma}{\sinh\Theta\cosh\Theta}\right) \to 0 \qquad \varphi_1 \sim -\frac{\kappa\sqrt{a}}{\cosh\Theta} \to 0 \qquad x \to \pm\infty.$$

For fixed x, we have the same formula when $t \to \pm\infty$.

As a function of x, u has a second-order pole and φ_1 has a first-order pole which locates at the same point $x = x_p(t)$ determined by the equation $\sinh\Theta\cosh\Theta - \kappa\gamma = 0$. Also, it is easy to see that $u(x,t)$ has two zeros and $\varphi_1(x,t)$ has one zero. The shape and the motion of $u(x,t)$ is the same as that described in [21].

Similarly, if we take $f = \cosh\Theta$, we can obtain the [C] one-negaton.

4.2. Two-negaton solution and multinegaton solutions

The [S] two-negaton solution of equation (2.1) with $n = 2$ and $\lambda_j = -\kappa_j^2 < 0, \kappa_j > 0$, $j = 1, 2$ is given by equation (2.13) with $N = 2$

$$f_j = \sinh\Theta_j \qquad \Theta_j = \kappa_j\big(x + x_j - 4\kappa_j^2 t\big) \qquad e_j(t) = a_j t + b_j \qquad j = 1, 2$$

$$\begin{aligned} W_1(f_1, f_2; e_1, e_2) = {} & (4\kappa_1\kappa_2)^{-1}(\kappa_1\gamma_1 - \sinh\Theta_1\cosh\Theta_1)(\kappa_2\gamma_2 - \sinh\Theta_2\cosh\Theta_2) \\ & - \big(\kappa_1^2 - \kappa_2^2\big)^{-2}(\kappa_2\sinh\Theta_1\cosh\Theta_2 - \kappa_1\sinh\Theta_2\cosh\Theta_1)^2 \end{aligned} \tag{4.3}$$

$$\begin{aligned} W_2(f_2, f_1; e_2) = {} & (2\kappa_2)^{-1}\sinh\Theta_1(\kappa_2\gamma_2 - \sinh\Theta_2\cosh\Theta_2) \\ & + \big(\kappa_1^2 - \kappa_2^2\big)^{-1}\sinh\Theta_2(\kappa_2\sinh\Theta_1\cosh\Theta_2 - \kappa_1\sinh\Theta_2\cosh\Theta_1) \end{aligned} \tag{4.4}$$

$$\begin{aligned} W_2(f_1, f_2; e_1) = {} & (2\kappa_1)^{-1}\sinh\Theta_2(\kappa_1\gamma_1 - \sinh\Theta_1\cosh\Theta_1) \\ & + \big(\kappa_1^2 - \kappa_2^2\big)^{-1}\sinh\Theta_1(\kappa_2\sinh\Theta_1\cosh\Theta_2 - \kappa_1\sinh\Theta_2\cosh\Theta_1) \end{aligned} \tag{4.5}$$

$$\gamma_j = x + \widetilde{x}_j - 12\kappa_j^2 t + 2a_j t + b_j \qquad \widetilde{x}_j = x_j + \kappa_j\partial_{\kappa_j}x_j(\kappa_j) \qquad \operatorname{Im} x_j = 0 \qquad j = 1, 2.$$

In the domain where $x + x_1 - 4\kappa_1^2 t$ is fixed and $t \to \pm\infty$, the asymptotic solution is

$$u = 2\partial_x^2\ \ln(\kappa_1\gamma_1 - \sinh\Theta_1\cosh\Theta_1)[1 + O(t^{-1})]$$

$$\varphi_1 = \frac{2\kappa_1\sqrt{a_1}\sinh\Theta_1}{\kappa_1\gamma_1 - \sinh\Theta_1\cosh\Theta_1}[1 + O(t^{-1})] \qquad \varphi_2 = O(t^{-1}).$$

When $x + x_2 - 4\kappa_2^2 t$ is fixed and $t \to \pm\infty$, we have a similar result for the asymptotic solution. This estimates show that, in the indicated domain, the leading term of the asymptotic [S] two-negaton solution is a standard [S] one-negaton solution. In other words, negatons are totally insensitive to the mutual collision, even without additional phase shifts in contrast to the solitons collision case.

Similarly we can construct the [C] two-negaton and [SC] two-negaton solutions and find the same property.

The [S] N-negaton solution of equation (2.1) with $n = N$ and $\lambda_j = -\kappa_j^2 < 0, \kappa_j > 0$, $j = 1, \ldots, N$ is given by equation (2.13) with $e_j = a_j t + b_j$

$$f_j = \sinh\Theta_j \qquad \Theta_j = \kappa_j\big(x + x_j - 4\kappa_j^2 t\big) \qquad \operatorname{Im} x_j = 0 \qquad j = 1, \ldots, N.$$

Analogously, we see that the [S] N-negaton solution at large time decays into the sum of N [S] free negatons.

5. Multisoliton–positon, multisoliton–negaton and multipositon–negaton solutions

Like the KdV equation, the KdVES also has multisoliton–positon, multisoliton–negaton and multipositon–negaton solutions. The N-positon M-soliton solutions of equation (2.1) with $n = N + M$ and $\lambda_j = \kappa_j^2 > 0$, $j = 1, \ldots, N$, $\lambda_{N+j} = -\kappa_{N+j}^2 < 0$, $j = 1, \ldots, M$ are given by equation (2.13) with N replaced by $N + M$ and

$$f_j = \sin\Theta_j \qquad \Theta_j = \kappa_j\big(x + x_j + 4\kappa_j^2 t\big) \qquad \kappa_j > 0 \qquad \operatorname{Im} x_j = 0$$

$$j = 1, \ldots, N \qquad f_{N+j} = \mathrm{e}^{\kappa_{N+j}(x - 4\kappa_{N+j}^2 t)} \qquad \kappa_{N+j} > 0 \qquad j = 1, \ldots, M.$$

The N-negaton M-soliton solution of equation (2.1) with $n = N + M$ and $\lambda_j = -\kappa_j^2 > 0$, $j = 1, \ldots, N + M$, is given by equation (2.13) with N replaced by $N + M$ and

$$f_j = \sinh\Theta_j \qquad \Theta_j = \kappa_j\big(x + x_j - 4\kappa_j^2 t\big) \qquad \kappa_j > 0 \qquad \operatorname{Im} x_j = 0$$

$$j = 1, \ldots, N \qquad f_{N+j} = \mathrm{e}^{\kappa_{N+j}(x - 4\kappa_{N+j}^2 t)} \qquad \kappa_{N+j} > 0 \qquad j = 1, \ldots, M.$$

The N-positon M-negaton solution of equation (2.1) with $n = N + M$ and $\lambda_j = \kappa_j^2 > 0$, $j = 1, \ldots, N$, $\lambda_{N+j} = -\kappa_{N+j}^2 < 0$, $j = 1, \ldots, M$ is given by equation (2.13) with N replaced by $N + M$ and $\operatorname{Im} x_j = 0$

$$f_j = \sin\Theta_j \qquad \Theta_j = \kappa_j\big(x + x_j + 4\kappa_j^2 t\big) \qquad \kappa_j > 0 \qquad j = 1, \ldots, N$$

$$f_{N+j} = \sinh\Theta_{N+j} \qquad \Theta_{N+j} = \kappa_{N+j}\big(x + x_{N+j} - 4\kappa_{N+j}^2 t\big) \qquad \kappa_{N+j} > 0 \qquad j = 1, \ldots, M.$$

We can analyse the interaction of the soliton and the positon, the soliton and the negaton, the positon and the negaton in a similar way as in [20]. We would like to point out that the results of the analysis will be almost the same as in [20] and we omit it.

6. Conclusions

We present N-times repeated GBDT with N arbitrary t-functions which provides non-auto-Bäcklund transformation between two KdV equations with n-degrees of sources and $(n + N)$-degrees of sources. This N-times repeated GBDT enables us to construct some general solutions with N arbitrary t-functions for the KdVES. By making a special choice of t-functions, we obtain the multisoliton, multinegaton, multipositon, multisoliton–positon, multisoliton–negaton and multipositon–negaton solutions for the KdVES. This method can be applied to other SESCSs.

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